ON THE CLOSED GRAPH THEOREM

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ABSTRACT

This is a study of the closed graph theorem for linear mappings from a locally convex space E into another F. First we look at the possible spaces E for fixed F. Next we fix such E and look at F. Finally we study the situation where $E=F$.

1. Given a locally convex space F, we ask for the locally convex spaces E such that every closed linear map from E into F is continuous. In general for such domain space E, the closed graph theorem holds with the range space F_1 more general than F, and such spaces E and F_1 are usually of different types. We then look at the spaces E such that the closed graph theorem holds for linear mappings of E into itself. Such play an important role in the study of the spectrum of a mapping on a locally convex space (see for example [11], p. 328, section 2). To such a space can be extended the result of Banach that, if a Banach space is the algebraic direct sum of two closed linear subspaces then it is their topological direct sum.

Our notation is pretty standard. Linear spaces are over a fixed field, either the real or complex numbers, denoted by R. For an index set Φ, R^{Φ} represents the product indexed by points from Φ of copies of R. Topological spaces are assumed Hausdorff.

2. Let F be a locally convex space. We call a locally convex space E an *F_t-space* if every closed linear map from E into F is continuous.

THEOREM 2.1. *Any inductive limit of F_t-spaces is an F_t-space.*

COROLLARY. *Any finite product of F_t-spaces is an F_t-space.*

However, an infinite product (even if countable) of F_t-spaces need not be an F_t-space. If for some infinite index set Φ, $(H,u)=R^{\Phi}$, and F is the linear space H

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under its finest locally convex topology, then R is an F_t-space. As the identity map $(H, u) \to F$ is closed but not continuous, $(H, u) = R^{\Phi}$ is not an F_t-space.

THEOREM 2.2. *If F is sequentially complete and has a fundamental sequence of bounded sets, then any product of F_t-spaces is an F_t-space.*

PROOF. Let (B_n) be a fundamental sequence of closed absolutely convex bounded subsets of F. If F_n is the linear span of B_n, let v_n be the normable topology on F_n with the sequence $(2^{-m}B_n: m = 1, 2, \cdots)$ of sets as a base of neighbourhoods. Using the sequential completeness of F, one shows that each (F_n, v_n) is complete. If (F, v) is the inductive limit of $((F_n, v_n))$ by the identity maps of (F_n, v_n) into F, then v is finer than the original topology of F.

Let f be a closed linear map from $E = X(E_\gamma: \gamma \in \Phi)$ into F, where each E_γ is an F_t-space. For some finite subset Φ_0 of Φ, if $\gamma \in \Phi_1 = \Phi \backslash \Phi_0$, then $f(E_\gamma) = 0$. Suppose no such Φ_0 exists. Then for a sequence $(\gamma_n: n = 1, 2, \cdots)$ of distinct members of Φ there is a sequence (x_{γ_n}) such that $x_{\gamma_n} \in E_{\gamma_n}$ and $f(x_{\gamma_n}) \neq 0$ for each n. Let $L = X(Rx_{\gamma_n}: n = 1, 2, \cdots)$. As the graph of the restriction f_1 of f to L is closed in $LX(F, v)$, it is continuous ([12], Theorem 2) and thus the set $(f(x_{\gamma_n}): n = 1, 2, \cdots)$ is bounded in F. For a positive integer N then, $(f(x_{\gamma_n}): n = 1, 2, \cdots) \subseteq B_N$. One shows from this that $f(L) \subseteq F_N$ and that the map $f_1: L \to (F_N, v_N)$ is continuous. As each $f(x_{\gamma_n}) \neq 0$, there is a sequence (λ_n) from R such that the set $(\lambda_n f(x_{\gamma_n})$: $n = 1, 2, \cdots)$ is not bounded in the normed space (F_N, v_N), though $((\lambda_n x_{\gamma_n}))$ is bounded in L and $f_1: L \to (F_N, v_N)$ is continuous. We then deduce that Φ_0 as above, exists. Thus $f(\Sigma\, E_\gamma: \gamma \in \Phi_1) = 0$. If f_2 is the restriction of f to

$$H = X(E_\gamma: \gamma \in \Phi_1),$$

then $f_2^{-1}(0)$ is closed in H, since the graph of f_2 is closed in HXF. As $\Sigma\,(E_\gamma: \gamma \in \Phi_1)$ $(\subseteq f_2^{-1}(0))$ is dense in H, we then see that $f(H) = 0$.

Since by the Corollary of Theorem 2.1, $X(E_\gamma: \gamma \in \Phi_0)$ is an F_t-space, the restriction of f to $X(E_\gamma: \gamma \in \Phi_0)$ is continuous. The map f is therefore continuous on E, and E is an F_t-space.

PROPOSITION 2.1. *Let E be a locally convex space and E_0 a linear subspace of codimension one. Then*

(i) *if E is an F_t-space so is E_0,*

(ii) *if E_0 is an F_t-space, so is E.*

PROOF. (i) Let f be a closed linear map from E_0 into F. The closure G_1 in EXF of the graph G of f is the graph of some linear extension f_1 of f, since if

$(0, y) \in G_1$, then $(0, y) \in G_1 \cap (E_0 X F) = G$ (because G is closed in $E_0 X F$), implying that $y = 0$. Either the domain of f_1 is E_0 or E. In the former case, if f_2 is any linear extension of $f_1 (= f)$ to E, the graph G_2 of f_2 can be put in the form $G_1 + H$, where H is a finite dimensional subspace of EXF. Thus the graph of f_2 is closed in EXF. As E is an F_t-space, f_2 is continuous. This implies that f is continuous and that E_0 is an F_t-space.

(ii) Either E_0 is closed or dense in E. If E_0 is closed in E, then E is topologically isomorphic to $E_0 X R$, and thus if E_0 is an F_t-space, so is E, since R is necessarily an F_t-space. Now suppose that E_0 is dense in E.

If $f: E \to F$ is a closed linear map, then the restriction f_0 of f to E_0 is continuous, being closed. By ([6], 5.5), the closure of the graph of f_0 in EXF, is the graph of a continuous linear extension f_1 of f. As the domain E_1 of f_1 is necessarily closed in E, $E_1 = E$. The graph of the map $f_1 - f$ is closed in EXF and therefore its null space (which contains E_0) is closed in E. Thus f_1, f coincide on E, f is continuous and E is an F_t-space.

Proposition 2.1 clearly extends by induction to the situation where E_0 is any subspace of finite codimension in E.

Theorem 2.3. *If F is as in Theorem 2.2, then any locally convex space is topologically isomorphic to a closed linear subspace of some F_t-space.*

Proof. For such F, let the topology v be as defined in the proof of Theorem 2.2. Any closed linear map from a Banach space into (F, v) is continuous ([12], Theorem 2). A Banach space is therefore an F_t-space for such F, and by Theorem 2.2, so is any product of Banach spaces.

Given a locally convex space E, there is a product H of Banach spaces such that E is (topologically isomorphic to) a subspace of H. By Proposition 2.1 (i), any subspace of H of codimension one is an F_t-space. Let E_1 be an algebraic supplement of E in H, and let $(e_\gamma: \gamma \in \Omega)$ be a Hamel basis for E_1. If for each λ in Ω, E_λ is the linear span of E and $(e_\gamma: \gamma \in \Omega,\ \gamma \neq \lambda)$, then E_λ is an F_t-space. Moreover, $E = \bigcap (E_\lambda: \lambda \in \Omega)$. One then shows using a technique of Y. Komura ([7], Theorem 1.1) that the map f from E into $X(E_\gamma: \gamma \in \Omega)$ defined as follows: for each x in E, $f(x) = (x_\alpha)$ where $x_\alpha = x$ for all α in Ω, is a topological isomorphism, and that $f(E)$ is closed in $X(E_\gamma: \gamma \in \Omega)$.

Corollary. *If F is as in Theorem 2.2 and has infinite dimension, then a closed subspace of an F_t-space need not be an F_t-space.*

PROOF. On such F there can be defined a Hausdorff locally convex topology v strictly coarser than the original topology of F. Clearly, (F, v) is not an F_t-space, but by Theorem 2.3, it can be embedded as a closed linear subspace of some F_t-space.

Given a locally convex space F, we say that a locally convex space F_1 is a *$D_r(F_t)$-space* if for each F_t-space E, any closed linear one-to-one map of F_1 onto E is open.

Clearly, a $D_r(F_t)$-space remains so for all coarser Hausdorff locally convex topologies.

THEOREM 2.4. *A locally convex space (F_1, v) is a $D_r(F_t)$-space if and only if for each F_t-space E, any closed linear map from E into (F_1, v) is continuous.*

PROOF. One part is obvious; now for the other. Let (F_1, v) be a $D_r(F_t)$-space and f a closed linear map from an F_t-space E into (F_1, v).

Since f is closed and linear, for some Hausdorff locally convex topology u say, on F_1 coarser than v, the map f from E into (F_1, u) is continuous. Let w be the finest locally convex topology on F_1 such that f is continuous. Thus $u \leqq w$ and one shows that since E is an F_t-space, so is (F_1, w). As $u \leqq w$ and (F_1, u) is Hausdorff, the graph of the identity map on F_1 is closed in $(F_1, u) X (F_1, w)$. Since $u \leqq v$, this graph is also closed in $(F_1, v) X (F_1, w)$. Because (F_1, v) is a $D_r(F_t)$-space and (F_1, w) is an F_t-space, the identity map from (F_1, v) onto (F_1, w) is open. Thus $v \leqq w$, and as the map f from E into (F_1, w) is continuous, it is also continuous when F_1 is considered under the coarser topology v.

If F is allowed to range over all Banach spaces, then the F_t-spaces are the t-spaces i.e. barrelled spaces (see [8], Theorem 2.2). Theorem 2.2 then shows (the well known fact) that a product of t-spaces is a t-space and Theorem 2.4 is a characterization of the range spaces for which the closed graph theorem holds for linear maps from t-spaces. F may also be allowed to range over all strict inductive limits of sequences of Banach spaces; the F_t-spaces are then the τ-spaces of [4]. All the above theory goes through in this case and similarly when F ranges over the strong duals of metrizable locally convex spaces. The generalized strict inductive limit of a sequence of Banach spaces has a fundamental sequence of bounded sets, but need not be complete. The technique of proof of Theorem 2.2 extends to this situation. Such F_t-spaces are looked at in [5], where an attempt is made to describe the topology of such a space in terms of subsets, in a manner analogous to that of a t-space.

Now, allow F vary over all generalized LF-spaces (see [6], p. 218, problem C). By ([12], Theorem 2), every inductive limit of second category locally convex spaces is an F_t-space, and by an application of ([8], Theorem 2.2), every F_t-space is a t-space. Let us look at ([6], p. 195, problem D). G^* is an F_t-space with a closed linear subspace H^0 which is not an F_t-space, since it is not a t-space. G/H is a generalized LF-space which is not complete. As G/H is a t-space which is not B_r-complete, there is a strictly coarser t-space topology v say, on G/H. As the identity map $(G/H, v) \to G/H$ is closed but not continuous, the t-space $(G/H, v)$ is not an F_t-space. Y. Komura ([7], p. 155) gave an example of a locally convex space which is not bornological but has a linear subspace of codimension one which is an inductive limit of Banach spaces. By an application of Proposition 2.1 (ii), we then see that an F_t-space need not be an inductive limit of normed linear spaces.

With F as above (i.e. a generalized LF-space), every generalized LF-space is a $D_r(F_t)$-space, and so is any B_r-complete locally convex space, since each F_t-space is a t-space. A sequentially complete locally convex space (E, u) with a fundamental sequence of bounded sets is necessarily a $D_r(F_t)$-space for, there is a topology v on E finer than u such that (E, v) is the generalized strict inductive limit of a sequence of Banach spaces (see the first paragraph of the proof of Theorem 2.2). In particular, the strong dual of any metrizable locally convex space is a $D_r(F_t)$-space.

In the discussion above, one might similarly allow F vary over all Souslin spaces (in the sense of [13]).

3. A locally convex space E is called a *closed graph space* or more shortly a *CG-space* if every closed linear map on E (i.e. every linear map $E \to E$ with graph closed in $E X E$) is continuous.

A B_r-complete t-space is a CG-space. In particular every Frechet space is a CG-space, and so is R^{Φ} for each index set Φ. Any generalized LF-space is a CG-space ([12], Theorem 2). By [13], the distribution spaces $\mathscr{E}, \mathscr{E}'$, $\mathscr{D}$, $\mathscr{D}'$, $\mathscr{S}$, $\mathscr{S}'$, $\mathscr{O}_M$, $\mathscr{O}'_M$, $\mathscr{O}_c$, $\mathscr{O}'_c$ are CG-spaces. A linear space is a CG-space under its finest locally convex topology, since every linear map on such a space is continuous.

A normed linear space may be a CG-space even if it is not complete. For if (E, u) is the sequence space $l^{\frac{1}{2}}$ and u^{00} is the finest locally convex topology on E coarser than u, then the graph of a closed linear map f on (E, u^{00}) is necessarily closed in $(E,u)X(E,u)$. Therefore f is continuous on (E,u) and we deduce from this that f is continuous on (E,u^{00}).

The dual of a Frechet space is B_r-complete ([12], p. 9) under the topology of compact convergence, but need not be a t-space.

THEOREM 3.1. *The dual of a Frechet-space is a CG-space under the topology of compact convergence.*

PROOF. Let E be a Frechet space with dual E'. Let $\sigma(E',E)$, $\tau(E',E)$ respectively be the weak and Mackey topologies on E' relative to E and v the topology on E' of compact convergence. Then $\sigma(E'E) \leqq v \leqq \tau(E'E)$. Let $f:(E',v) \to (E',v)$ be a closed linear map. The graph of f is necessarily closed in $(E',\sigma(E'E))X(E',\sigma(E,'E))$.

If (U_n) is a fundamental sequence of closed absolutely convex neighbourhoods of the origin in E, then their polar sets (U_n^0) in E' are σ (E',E)-compact and $E' = \bigcup(E_n': n = 1,2,\cdots)$, where E_n' is the linear span of U_n^0. For each n, the sets $(2^{-m}U_n^0: m = 1,2,\cdots)$ form a base of neighbourhoods in a normable topology u_n on E_n' and each (E_n', u_n) is complete. Let (E', u) be the inductive limit of the Banach spaces (E_n', u_n) by the identity maps $(E_n', u_n) \to E'$. The graph of the restriction of f to each (E_n', u_n) is closed in $(E_n', u_n)X(E', u)$ and therefore each restriction of $f:(E_n', u_n) \to (E', u)$ is continuous and hence bounded. For each n then there is $m(n)$ such that

$$f(U_n^0) \subseteq U_{m(n)}^0.$$

Now consider the map f on $(E,'\sigma(E',E))$. The graph G_n of the restriction f_n of f to U_n^0 is closed in the compact Hausdorff space

$$W_n = U_n^0 X\, U_{m(n)}^0.$$

Let P_n, $P_{m(n)}$ be the projection maps of W_n onto U_n^0, $U_{m(n)}^0$ respectively. The restriction P_{G_n} of P_n to G_n is continuous one-to-one and must then be a homeomorphism, since G_n is compact. As $f_n = P_{m(n)} \circ P_{G_n}^{-1}$, each f_n is continuous.

The finest locally convex topology on E' coinciding with $\sigma(E',E)$ on each U_n^0 is v, and the map $f:(E',v) \to (E',\sigma(E',E))$ is continuous (see [2], p. 16, Ex. B). This implies that $f:(E', v) \to (E', v)$ is continuous and that (E',v) is a CG-space.

COROLLARY. *The dual of a Frechet space is a CG-space under the Mackey topology.*

PROOF. In the notation of the proof of Theorem 3.1, if h is a closed linear map on $(E', \tau(E', E))$, the graph of h is necessarily closed in $(E', v) X (E', v)$. By Theorem 3.1, the map h is continuous on (E', v). This implies that h is continuous on $(E', \tau(E', E))$ (see [6], 21.6).

By the technique above one can show that the closed graph theorem holds for linear maps from the Mackey dual of a Frechet space to another.

PROPOSITION 3.1 *Let E be a linear space.*

(i) *If u, v are respectively the weak and Mackey topologies on E with the same dual, then (E,u) is a CG-space if and only if (E,v) is a CG-space.*

(ii) *If $(u_\alpha: \alpha \in \psi)$ is a set of topologies on E each making it a CG-space and the finest locally convex topology u on E coarser than all the u_α, is Hausdorff then (E,u) is a CG-space.*

(iii) *If $u < v$ are topologies on E such that (E,u), (E,v) are CG-spaces, then $(E,u) X (E,v)$ is not a CG-space.*

PROOF. (i) By ([6], 21.4), a linear map $f: (E,u) \to (E,u)$ is continuous if and only if it is continuous as a map $(E,v) \to (E,v)$.
(i) now follows since the graph of f is closed in $(E,u) X (E,u)$ if and only if it is closed in $(E,v) X (E,v)$ ([6], 17.1).

(ii) A closed linear map f on (E,u) is also a closed map on (E,u_α) and this implies that for all α in ψ, the map $f: (E,u_\alpha) \to (E,u)$ is continuous. As (E,u) is the inductive limit of (E,u_α) by the identity maps $(E,u_\alpha) \to E$, we deduce that the map f is continuous on (E,u) and that (E,u) is a CG-space.

(iii) Let $f(x,y) = (y,x)$ be a map on $(F,w) = (E,u) X (E,v)$. As $f: (F,w) \to (E,u) X (E,u)$ is continuous, its graph is closed in $(F,w) X (F,w)$. It is not difficult to see that the map f on (F,w) is not continuous and thus (F,w) is not a CG-space.

If (E,u) is a Frechet Montel space of infinite dimension and v is the finest locally convex topology on E, then the product space $(E,u) X (E,v)$ is a complete bornological Montel space, which by Proposition 3.1 (iii) is not a CG-space. If instead we consider the situation where (E,v) is a Banach space of infinite dimension, and u is the weak topology associated with v, then by looking at $(E,u) X (E,v)$, we deduce from Proposition 3.1 (i) and (iii) that a locally convex space (F, w_1) may be a CG-space whereas (F, w_2) is not, even if $w_1 < w_2$ and $(F, w_1)' = (F, w_2)'$. Also in this case, (F, w_1), (F, w_2) could be CG-spaces and for some locally convex topology between w_1 and w_2, F is not a CG-space. These indicate that unlike the situation of B_r-complete spaces, a successful study of CG-spaces by means of duality theory of locally convex spaces can hardly be expected.

Lemma 2 of [3] is still valid with "Banach spaces" replaced by "CG-spaces". As an example, let $(r_i: i = 1,2,\cdots)$ be a strictly increasing sequence of real numbers such that for some k and all i, $1 \leqq r_i < k$. If (E_i, u_i) is the sequence space l^{r_i}, then

$$E_1 \subset E_2 \subset \cdots \subset E_i \subset \cdots \subset l^k.$$

Put $E = \bigcup_i E_i$ and let (E,u) be the inductive limit of $((E_i, u_i))$ by the identity maps $(E_i, u_i) \to E$. The topology u is finer than that induced on E by l^k and therefore (E,u) is Hausdorff. (E,u), l^k and all (E_i, u_i) are CG-spaces. Therefore if f is a closed linear map on

(a) l^k such that $f(E) \subseteq E$ then the restriction of f to (E,u) is continuous,

(b) (E,u) such that for some N, $f(E_N) \subseteq E_N$, then the restriction of f to (E_N, u_N) is continuous.

A closed linear subspace E_1 of a Hilbert space E has a closed algebraic supplement in E; similarly if for index set Φ, $E = R^{\Phi}$ or if E is a linear space under its finest locally convex topology. But this does not hold for arbitrary Banach spaces. For, let $1 < p < \infty$, $p \neq 2$ and let $(E,u) = l^p$. Then there is a closed linear subspace E_1 of (E,u) which has no closed algebraic supplement. For such E_1, there is a u-closed linear subspace E_2 of E such that $E_1 \cap E_2 = (0)$ and $E_0 = E_1 + E_2$ is dense in (E,u) ([10], p. 77).

LEMMA 3.1. *If a locally convex space E is the algebraic direct sum of two closed linear subspaces E_1, E_2 then for each $i = 1,2$, the graph of the projection map $P_i: E \to E_i$ is closed in $E X E$. E is the topological direct sum of E_1, E_2 if and only if the maps P_1, P_2 are continuous.*

THEOREM 3.2. *If a CG-space is the algebraic direct sum of two closed linear subspaces E_1, E_2, then E is the topological direct sum of E_1, E_2. Cf ([9], Corollary 2.1).*

COROLLARY. *Let E be a CG-space which is complete. If E_1, E_2 are closed linear subspaces of E such that $E_1 \cap E_2 = (0)$ and $E_0 = E_1 + E_2$ is dense in E, then under the induced topology, E_0 is the topological direct sum of E_1, E_2 if and only if $E_0 = E$.*

PROOF. E_1, E_2 are complete and so is the dense subspace E_0 of E if E_0 is the topological direct sum of E_1, E_2. The converse assertion follows immediately from the theorem.

Thus by the Corollary above, the normed linear space E_0 referred to in the paragraph preceding Lemma 3.1, is not a CG-space.

By Proposition 3.1. (iii), the topological direct sum of two CG-spaces need not be a CG-space. B_r-complete t-spaces form an important class of CG-spaces; it is yet unknown if the topological direct sum of two such spaces is B_r-complete.

LEMMA 3.2. *Let (E,u) be the topological direct sum of two B_r-complete locally convex spaces (E_1,u_1) and (E_2,u_2). If there is a Hausdorff t-space topology v on E strictly coarser than u, then v induces topologies v_1,v_2 respectively on E_1, E_2 such that $v_1 < u_1$ and $v_2 < u_2$.*

PROOF. If $u_1 = v_1$, then (E_1,v_1) is complete, being B_r-complete. Thus E_1 is closed in (E,v); let $k\colon E \to E/E_1$ be the quotient map. The restriction $k_2\colon (E_2,v_2) \to (E,v)/E_1$ of k to E_2 is continuous one-to-one and onto. As (E_2,u_2) is B_r-complete and $(E,v)/E_1$ is a Hausdorff t-space, the map k_2 is open and one uses this to show that $u = v$.

THEOREM 3.3. *If (E_1,u_1) is a B_r-complete t-space and for some index set Φ, $(E_2,u_2) = R^{\Phi}$, then the topological direct sum (E,u) of (E_1,u_1) and (E_2,u_2) is B_r-complete.*

PROOF. The space (E,u) is a t-space. Since there is no Hausdorff locally convex topology on E_2 strictly coarser than u_2, we deduce from Lemma 3.2 that on E there is no Hausdorff t-space topology strictly coarser than u. This implies that the t-space (E,u) is B_r-complete.

Cf.([1], Proposition 5 and [9], Theorem 2.1).

Let us conclude with a remark that throws light on Theorem 5 of [14]. As the Banach sequence spaces l^2, c_0 are algebraically isomorphic, there are two Banach space topologies v_1, v_2 on some linear space H such that $l^2(c_0)$ is topologically isomorphic to (H_1,v_1) $((H,v_2))$. Let u_1,u_2 be distinct topologies each making a linear space E a B_r-complete t-space. If u_3 is the finest locally convex (= finest linear) topology on E coarser than both u_1 and u_2, then (E,u_3) is not Hausdorff. For otherwise, the graph of the identity map on E would be closed in (E,u_1) $X(E,u_2)$ implying that $u_1 = u_2$.

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